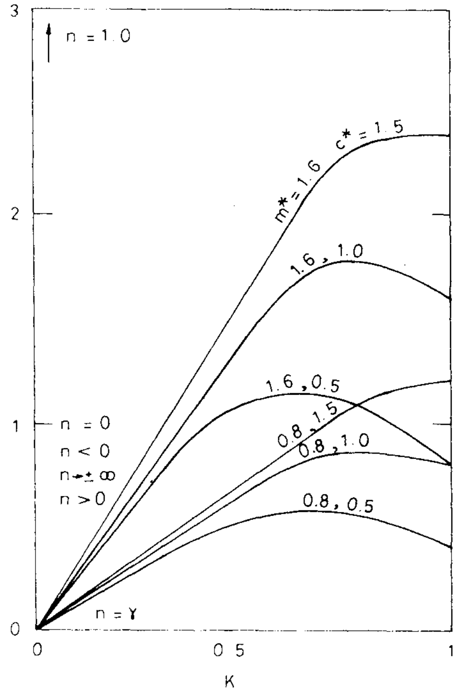

Fig. 1 Gas polytropic specific heat ratio vs similarity parameter K for different values of m^* and c^* at $Pr = 0.71$, $\gamma = 1.4$.

isochore, isentropic, and isotherm changes of state, respectively. The value of (c_n/c_p), and hence n, is a function of m^*, c^* K, γ, and Pr. Taking $\gamma = 1.4$ and $Pr = 0.71$, (c_n/c_p) is plotted in Fig. 1 as a function of K. It is shown that depending on m^*, c^*, and K, (c_n/c_p) can have all of the values corresponding to the above four changes of state or any intermediate values.

Now defining a gas stagnation temperature $T^{0\prime} = T' + u'^2/(2c_p)$, it is possible to show from Eqs. (1a-c) that $T^{0\prime} = T_c^0 + C_3 x^2$, where

$$C_3 = m^* C_1^2 (Lc^* - K^2) / [c_p (m^* c^* L + 1)] \tag{5}$$

It is, therefore, shown that stagnation temperature in the flow direction increases with x^2. Finally, because of constant n during expansion, one can calculate various flow properties in the nozzle from closed-form solutions, which at the throat become[6]

$$T_c^0/T_t' = 2/(n+1);\; p_c^0/p_t' = (T_c^0/T_t')^{n/(n-1)};\quad M_t = \sqrt{n/\gamma} \tag{6}$$

References

[1] Thomas, A.N., "New Generation Ramjets—A Promising Future," *Astronautics & Aeronautics*, Vol. 18, June 1980, pp. 36-41, 71.

[2] Billig, F.S., Waltrap, P.J., and Stockbridge, R.D., "Integral-Rocket Dual-Combustion Ramjets: A New Propulsion Concept," *Journal of Spacecraft and Rockets*, Vol. 17, Sept.-Oct. 1980, pp. 416-424.

[3] Hassan, H.A., "Exact Solutions of Gas-Particle Nozzle Flows," *AIAA Journal*, Vol. 2, Feb. 1964, pp. 395-396.

[4] Soo, S.L., *Fluid Dynamics of Multiphase Systems*, Blaisdell Publishing Co., Waltham, Mass, 1967.

[5] Rudinger, G., "Gas-Particle Flow in Convergent Nozzles at High Loading Ratio," *AIAA Journal*, Vol. 8, July 1970, pp. 1288-1294.

[6] Bose, T.K., "Effect of Heat Transfer in a Converging-Diverging Nozzle," *Journal of Spacecraft and Rockets*, Vol. 4, March 1967, pp. 401-402.

[7] Bose, T.K., "Thermodynamic Expansion Processes for Argon Plasma in a Convergent-Divergent Nozzle," *Journal of Spacecraft and Rockets*, Vol. 10, Sept. 1973, pp. 613-615.

AIAA 82-4178

Geodetic Latitude of a Point in Space

Terrence W. Barbee*
Space Applications Corporation, Irvine, Calif.

Introduction

THE problem of computing the geodetic latitude of an arbitrary point above a reference ellipsoid has been examined by many people.[1] However, these papers were primarily concerned with obtaining approximations to geodetic latitude rather than with the exact solution. This paper presents the exact closed form solution to the problem. It is possible, however, that the ponderous form of the exact solution may not prove to be of practical use in situations where rough approximation may suffice. This Note represents the first known attempt at the exact algebraic solution.

Analytical Development

Consider the problem of finding the geodetic latitude of an arbitrary point P lying above the reference ellipsoid

$$\frac{x^2+y^2}{a_e^2} + \frac{z^2}{b_e^2} = 1$$

where $a_e = 6378.145$ km and $b_e = 6356.759$ km. Owing to polar symmetry it suffices to restrict analysis to a meridian plane containing the point P. Let (r,z) denote the abscissa and ordinate respectively in the meridian plane so that x, y, and r are related by

$$r = \sqrt{x^2+y^2}$$

and the ellipse under consideration is given in parametric form by

$$r = a_e \cos t$$

$$z = b_e \sin t \qquad -\pi/2 \le t \le \pi/2 \tag{1}$$

If P has Cartesian coordinates, $P = (x_0, y_0, z_0)$, then the meridian coordinates of P will be (r_0, z_0), where $r_0 = (x_0^2 + y_0^2)^{1/2}$. Let θ denote the geodetic latitude of P. The geodetic normal through P intersects the ellipse [Eq. (1)] at some point $Q = (a_e \cos t,\ b_e \sin t)$, as shown in Fig. 1. The fundamental equality

$$\tan\theta = (a_e/b_e)\tan t \tag{2}$$

relates θ to t in a one-to-one manner for $t \epsilon [-\pi/2, \pi/2]$, so that in order to determine θ it will suffice to find the value of t which defines the point Q. The two quantities, $\tan\theta$ and $(z_0 - b_e \sin t)/(r_0 - a_e \cos t)$, each represent the slope of the geodetic normal through P so that by Eq. (2),

$$z_0 - b_e \sin t = (a_e/b_e)(\tan t)(r_0 - a_e \cos t)$$

which can be written as

$$\sin t\ \cos t + A\ \cos t = B\ \sin t \tag{3}$$

where A and B are given by

$$A = \frac{b_e z_0}{a_e^2 - b_e^2} \qquad B = \frac{a_e r_0}{a_e^2 - b_e^2} \tag{4}$$

Received July 31, 1981; revision received Dec. 1, 1981.

*Member of Technical Staff.

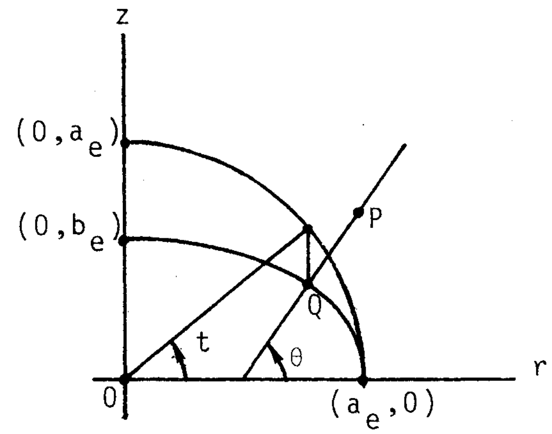

Fig. 1 Reference ellipse and circumscribed circle.

Squaring both sides of Eq. (3) results in

$$x^4+ax^3+bx^2+cx+d=0 \tag{5}$$

where $x=\sin t$ and a, b, c, d are given by

$$a=2A \qquad b=A^2+B^2-1 \qquad c=-2A \qquad d=-A^2 \tag{6}$$

Of course, $\sin t$ is one solution of Eq. (5) and the objective here is to exhibit $\sin t$ as an explicit function of A and B. To this end Ferrari's classical solution of the quartic, as well as Tartaglia's solution of the general cubic are used. To simplify the analysis, let $z_0>0$ and $r_0>0$. These restrictions will be removed later. The resolvent cubic equation relative to the quartic [Eq. (5)] is given by[2]

$$y^3+py^2+qy+r=0 \tag{7}$$

where

$$p=-(A^2+B^2-1) \qquad q=0 \qquad r=-4A^2B^2$$

because the coordinates of P satisfy $r_0^2+z_0^2 \geq b_e^2$, it is easy to see that $p<0$, and thus the resolvent cubic Eq. (7) experiences only one change of sign in its coefficients. Hence by a well-known theorem of Descartes,[2] Eq. (7) has exactly one positive root. If will be necessary to find this root in order to solve Eq. (5). If α and β are defined by

$$\alpha=-1/3(A^2+B^2-1)^2$$

$$\beta=-(2/27)(A^2+B^2-1)^3-4A^2B^2$$

then the three roots of Eq. (7) are given by[2]

$$y=S+T-\frac{p}{3}$$

$$x=-\frac{S+T}{2} \pm \frac{S-T}{2}\sqrt{-3}-\frac{p}{3} \tag{8}$$

where

$$\left\{\begin{matrix} S \\ T \end{matrix}\right\}=\left[-\frac{\beta}{2} \pm \sqrt{\frac{\beta^2}{4}+\frac{\alpha^3}{27}}\right]^{1/3}$$

A simple analysis shows that $\beta^2/4+\alpha^3/27>0$ and it easily follows that the first root in Eq. (8), $y=S+T-p/3$, is the (only) positive root of Eq. (7). Define a quantity R in terms of this root by

$$R=[a^2/4-b+y]^{1/2}=\sqrt{y+1-B^2} \tag{9}$$

Now R is nonzero, for otherwise $y=B^2-1$ would satisfy Eq. (7). Substitution of this value of y into Eq. (7) leads to $A^2(B^2+1)^2=0$, which would imply that $z_0=0$, contradicting the earlier assumption that $z_0>0$. The four roots of Eq. (5) can now be exhibited as

$$x=-a/4+R/2 \pm D/2$$
$$x=-a/4-R/2 \pm E/2 \tag{10}$$

where D and E are given by

$$\left\{\begin{matrix} D \\ E \end{matrix}\right\}=\left[\frac{3a^2}{4}-R^2-2b \pm \frac{4ab-8c-a^3}{4R}\right]^{1/2} \tag{11}$$

and $R \neq 0$ is given by Eq. (9).[2] Because $z_0>0$, so $A>0$ and the coefficients [Eq. (6)] therefore satisfy

$$a>0 \qquad b>0 \qquad c<0 \qquad d<0$$

Again by Descartes' theorem it follows that one and only one of the roots in Eq. (10) is positive, and it is this root x which of course satisfies $x=\sin t$. In order to pick this x from the four possibilities given in Eq. (10) it is necessary to prove that

1) $4ab-8c-a^3>0$,
2) R in Eq. (9) is real and positive.

Point 1 easily follows from the fact that $A>0$. To prove point 2, the following, not so obvious fact is needed:

$$x^4+ax^3+bx^2+cx+d=0$$

if and only if

$$(x^2+\tfrac{1}{2}ax+\tfrac{1}{2}y)^2=R^2\left[x+\frac{\tfrac{1}{2}ay-c}{2R^2}\right]^2 \tag{12}$$

This condition arises from Ferrari's original solution of the general quartic, given in 1545.[2] Now the positive root x must satisfy Eq. (12). But then $x>0$, $a>0$, and $y>0$ means that the left side of Eq. (12) is real and positive. R^2 is therefore positive, and point 2 is established. From points 1 and 2 it follows that D and E given by Eq. (11) satisfy $D^2>E^2$. Now if D were imaginary then $D^2<0$ and hence $E^2<0$, so that both D and E would be imaginary. This would mean that all four roots in Eq. (10) would be complex (not real), an impossibility. Therefore D must be real and positive by its very definition in Eq. (11). Regardless of whether E is real or imaginary, the inequality $D^2>E^2$ shows that the *largest* real root appearing in Eq. (10) must be

$$x=-a/4+R/2+D/2 \tag{13}$$

This value of x must be the positive root which satisfies $x=\sin t$.

The geodetic latitude of $P=(r_0,z_0)$ can now be computed in terms of x in Eq. (13) by writing Eq. (2) as

$$\theta=\tan^{-1}\left[\frac{a_e}{b_e}\frac{x}{\sqrt{1-x^2}}\right]$$

One can also remove the restriction $z_0>0$ by using the Heavyside multiplier $|z_0|/z_0$. That is, θ can be written as

$$\theta=\frac{|z_0|}{z_0}\tan^{-1}\left[\frac{a_e}{b_e}\frac{|x|}{\sqrt{1-x^2}}\right] \qquad z_0 \neq 0$$

$$=0 \qquad \text{otherwise}$$

With x given by Eq. (13), and R and D given by Eqs. (9) and (11), respectively, a closed form expression for θ can be given which involves only the parameters A and B. The desired expression, after some algebraic manipulation, is

$$\theta = \frac{|z_0|}{z_0} \tan^{-1}\left[\frac{a_e}{b_e}\frac{U-|A|}{\sqrt{4-(U-|A|)^2}}\right] \qquad z_0 \neq 0$$

$$= 0 \qquad z_0 = 0 \qquad (14)$$

where

$$U = \left[A^2 - B^2 + 1 - y + \frac{2|A|(B^2+1)}{\sqrt{y+1-B^2}}\right]^{1/2} + \sqrt{y+1-B^2}, \qquad (15)$$

$$y = [(1/27)(A^2+B^2-1)^3 + 2A^2B^2$$
$$+ \sqrt{(4/27)A^2B^2(A^2+B^2-1)^3 + 4A^4B^4}]^{1/3}$$
$$+ [(1/27)(A^2+B^2-1)^3 + 2A^2B^2$$
$$- \sqrt{(4/27)A^2B^2(A^2+B^2-1)^3 + 4A^4B^4}]^{1/3}$$
$$+ 1/3(A^2+B^2-1)$$

with

$$A = \frac{b_e z_0}{a_e^2 - b_e^2} \quad \text{and} \quad B = \frac{a_e r_0}{a_e^2 - b_e^2}$$

Note that the above equation for θ is actually valid for $r_0 = 0$, i.e., for P on the polar axis; for in this case we find that $B = 0$, $y = A^2 - 1$, $U = 2 + |A|$, and so $|\theta| = \pi/2$.

Example

Using Fig. 1 it is quite easy to compute the geodetic altitude of P. Letting H denote this altitude, then

$$H = (|z_0| - b_e \sin t)\csc\theta$$

Since θ and t are related by Eq. (2) this last expression can be written in terms of t as

$$H = \left[\frac{|z_0| - b_e \sin t}{\sin t}\right][(1-e^2) + e^2\sin^2 t]^{1/2}$$

where e is the eccentricity of the ellipse [Eq. (1)]. In terms of U in Eq. (15)

$$H = \left[\frac{2|z_0| - b_e(U-|A|)}{U-|A|}\right]\left[(1-e^2) + \frac{e^2}{4}(U-|A|)^2\right]^{1/2}$$

$$= r_0 - a_e \qquad \begin{matrix} z_0 \neq 0 \\ z_0 = 0 \end{matrix} \qquad (16)$$

Acknowledgments

Acknowledgment is given to Audrey Gioiello and Corine Turek for their help in the preparation of this paper. Space Applications Corporation, which provided the necessary funding for this Note, is also acknowledged.

References

[1] Berger, W.J. and Ricupito, J.R., "Geodetic Latitude and Altitude of a Satellite," *Journal of the American Rocket Society,* Vol. 1, Sept. 1960, pp. 901-902.

[2] Hart, W.L., *College Algebra,* D.C. Heath and Company, Boston, Mass., 1926, pp. 216-218, 236-239.

AIAA 82-4179

Gas Gun Study of Selected Buffers for Spall Fracture Reduction in Missile Materials

Willis Mock Jr.* and William H. Holt*
Naval Surface Weapons Center, Dahlgren, Va.

I. Introduction

A MISSILE warhead is usually located inside the outer skin or shroud part of a missile structure. After warhead detonation, the warhead fragments must penetrate the shroud to reach the target. The interaction of the fragments and shroud may cause decreased fragment velocity and fragment breakup. Recently a series of gas gun experiments was performed to study the effect of placing a polyethylene buffer material on the inside surface of the shroud to reduce fragment breakup.[1] The polyethylene thicknesses ranged from 1 to 9 mm. In the present investigation, 11 additional buffer materials have been studied. Experiments were performed for buffers in the 4-mm-thickness range. Composite specimens that simulated the shroud were impacted by steel disks. The impactor disks were soft recovered, sectioned, polished, and examined for fracture damage.

II. Experimental Techniques

A schematic of the muzzle region of the gas gun[2] with a target assembly containing a buffered composite specimen is shown in Fig. 1. A composite specimen consists of a wire harness layer, a 7075-T6 aluminum layer, a honeycomb layer, and another 7075-T6 aluminum layer. The projectile velocity is measured at impact with the three velocity pins. The steel impactor disk is supported only near its edge so the disk back surface is free over most of its area. The barrel is evacuated to minimize gas cushion effects on impact.

The impactor disks were fabricated from AISI C 1026 hot-rolled seamless tubing. The manufacturer's mechanical properties are 300-MPa yield strength, 540-MPa ultimate tensile strength, and 28% elongation. The measured impactor density was 7.83 Mg/m^3. The wire harness layer consisted of a series of parallel wires sealed between two thin plastic sheets. The honeycomb layer consisted of a plastic honeycomb material with a 0.072-Mg/m^3 density. Fast-setting epoxy was used to attach the four layers of a composite specimen together.

A gas gun shot was fired for each of the following buffer materials: Min-K 2000 molded insulation, polyrubber, polyurethane foam, silicone rubber, nylon, polyurethane, butyl rubber, polyester, neoprene, Micarta, and Melmac. The density of these materials ranges from 0.29 to 1.49 Mg/m^3. The average buffer thickness is 3.90 mm. The average impactor and specimen thicknesses are 9.1 and 9.4 mm, respectively. Average thickness values for the four specimen layers are as follows: 1.63-mm-thick wire harness layer, 1.61-mm-thick 7075-T6 aluminum layer, 5.41-mm-thick honeycomb layer, and 0.76-mm-thick 7075-T6 aluminum layer. The average diameter of the impactors and composite specimens is 29 mm.

A buffered specimen was secured inside a Lucite target holder with epoxy paste. The steel disks were soft recovered after impact to minimize any unintentional damage. The soft

Received Dec. 8, 1981; revision received Feb. 11, 1982.

*Research Physicist, Weapons Systems Department.